Algebraic Curve Segments

This report presents a number of algebraically defined curve segments which are useful for modeling the geometry of aerospace vehicles and for fitting engineering data.

The coefficients defining these curve segments can be determined by simple linear equations which are easily solved by standard elimination techniques.

The system of equations that determine the coefficients of the curve fit is simplified by re-writing the general equation so that the graph of the curve passes through the origin. This new equation will be referred to as the relative form of the general equation.

The original end points (X_1, Y_1) and (X_2, Y_2) are translated to become $(0,0)$ and $(\tilde{X}, \tilde{Y})$. The end point tangents T_1 and T_2 remain unchanged.

The system of equations that determine the coefficients of the algebraic curve segments are as follows:

1)	$Y = f(X)$	The relative form of the general equation which satisfies the constraint $0 = f(0)$.
2)	$T = f'(X)$	The tangent equation.
3)	$T_1 = f'(0)$	The initial tangent condition.
4)	$T_2 = f'(\tilde{X})$	The final tangent condition.
5)	$\tilde{Y} = f(\tilde{X})$	The final point condition.

Equations 3, 4, 5 and the constraint equation $0 = f(0)$ provide enough information to determine a function of three independent coefficients or parameters $Y = f(X, A, B, C)$.

If a two parameter function $Y = f(X, A, B)$ is used for the curve fit then equations 3, 4 and 5 will yield equations for A and B along with a compatibility equation which connects the values of $\tilde{X}, \tilde{Y}, T_1$ and T_2.

Algebraic Curve Fitting

Two Points and two slopes

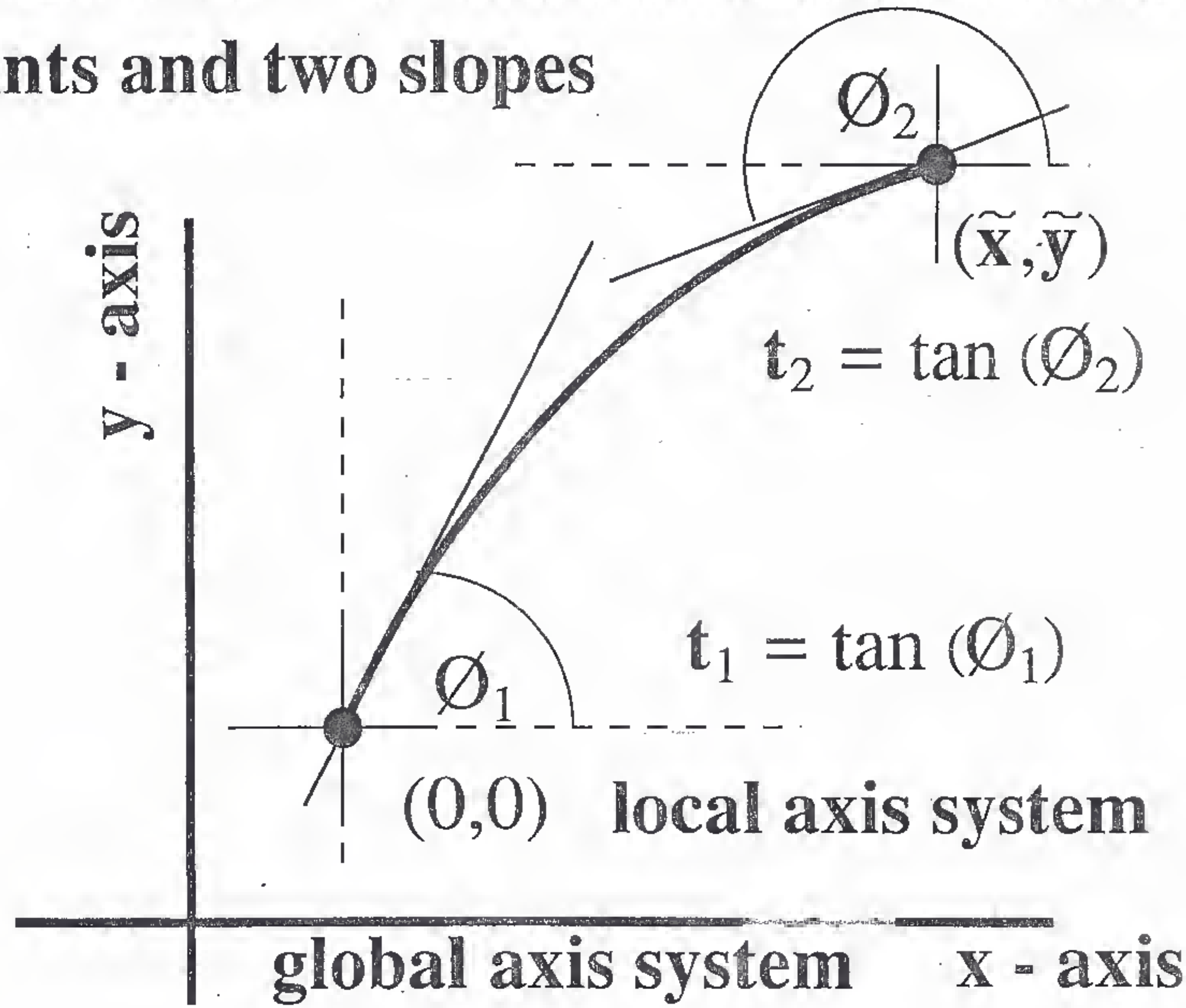

Table (1) Algebraic Fits for Two Parameter Equations

$$y = f(x, a, b)$$

Name	X-Axis Parabola	Y-Axis Parabola	Circle
General Form	$(y-k)^2 = 2p(x-h)$	$y = \frac{p}{2}(x-h)^2 + k$	$(x-h)^2 + (y-k)^2 = R^2$
Relative Form	$y^2 = ax + by$	$y = ax^2 + bx$	$x^2 + y^2 = ax + by$
Tangent Equation	$t = \frac{a}{2y - b}$	$t = 2ax + b$	$t = \frac{a - 2x}{2y - b}$
Compatibility	$\frac{\overline{y}}{\overline{x}} = \frac{2t_1t_2}{t_1 + t_2}$	$\frac{\overline{y}}{\overline{x}} = \frac{t_1 + t_2}{2}$	$\overline{y}^2 + \frac{2(1 - t_1 t_2)}{t_1 + t_2}\overline{xy} - \overline{x}^2 = 0$
"B" Equation	$b = \frac{\overline{y}^2}{\overline{y} - \overline{x}\, t_1} = \frac{2\overline{y}\, t_2}{t_2 - t_1}$	$b = t_1$	$b = \frac{\overline{x}^2 + \overline{y}^2}{\overline{y} - \overline{x}\, t_1} = \frac{2(\overline{y}t_2 + \overline{x})}{t_2 - t_1}$
"A" Equation	$a = -b\, t_1$	$a = \frac{t_2 - t_1}{2\overline{x}} = \frac{\overline{y} - \overline{x}\, t_1}{\overline{x}^2}$	$a = -bt_1$
Comments	Symmetric to x - axis	Symmetric to y- axis	

Table (2) Algebraic Fits for Two Parameter Equations (Continued)

$$y = f(x, a, b)$$

Name	Rectangular Hyperbola	Exponential	Logarithmic
General Form	$(x-h)(y-k) = R$	$y - k = a(1 - e^{bx})$	$y - k = a \ln(bx + 1)$
Relative Form	$xy = ax + by$	$y = a(1 - e^{bx})$	$y = a \ln(bx + 1)$
Tangent Equation	$t = \frac{a - y}{x - b}$	$t = -a b e^{bx}$	$t = \frac{a b}{bx + 1}$
Compatibility	$\bar{y}^2 + \bar{x}^2 t_1 t_2 = 0$	$\frac{\bar{y}}{\bar{x}} = \frac{t_2 - t_1}{\ln(t_2 / t_1)}$	$\frac{\bar{y}}{\bar{x}} = \frac{t_1 t_2}{t_1 - t_2} \ln(t_1 / t_2)$
"B" Equation	$b = \frac{\bar{x}\,\bar{y}}{\bar{y} - \bar{x} t_1} = \frac{\bar{x} t_2 + \bar{y}}{t_2 - t_1}$	$b = \frac{t_2 - t_1}{\bar{y}} = \frac{\ln(t_1 / t_2)}{\bar{x}}$	$b = \frac{t_1 - t_2}{\bar{y} t_2} = \frac{\ln(t_1 / t_2)}{\bar{y}} t_1$
"A" Equation	$a = -bt_1$	$a = -\frac{t_1}{b}$	$a = \frac{t_1}{b}$
Comments	$t_1 t_2 > 0 \quad t_1 \neq t_2$	$t_1 t_2 > 0 \quad t_1 \neq t_2$	$t_1 t_2 > 0 \quad t_1 \neq t_2$

Table (3) Algebraic Fits for Three Parameter Equations

$$y = f(x, a, b, c)$$

Name	Rotated x-axis Parabola	Rotated y-axis Parabola	Ellipse or Hyperbola
General Form	$(y-k)^2 = 2p(x-h) + 2dxy$	$y - k = \frac{p}{2}(x-h)^2 + dxy$	$(x-h)^2 \pm (y-k)^2 = R^2$
Relative Form	$y^2 = axy + bx + cy$	$y = ax^2 + bxy + cy$	$y^2 = ax^2 + bx + cy$
Tangent Equation	$t = \frac{ay + b}{2y - ax - c}$	$t = \frac{2ax + by + c}{1 - bx}$	$t = \frac{2ax + b}{2y - c}$
"C" Equation	$c = \frac{(\bar{x} t_2 - \bar{y})\,\bar{y}^2}{\bar{x}^2 t_1 t_2 - \bar{y}^2}$	$c = t_1$	$c = \frac{2\bar{y}^2 - 2\overline{xy}\, t_2}{2\bar{y} - \bar{x}(t_1 + t_2)}$
"B" Equation	$b = -c\, t_1$	$b = \frac{\bar{x}(t_1 + t_2) - 2\bar{y}}{\bar{x}^2 t_2 - \overline{xy}}$	$b = -c\, t_1$
"A" Equation	$a = \frac{\bar{y}(2\bar{x} t_1 t_2 - \bar{y}(t_1 + t_2))}{\bar{x}^2(t_1 t_2) - \bar{y}^2}$	$a = \frac{\bar{y}^2 - \bar{x}^2 t_1 t_2}{(\bar{x} t_2 - \bar{y})\,\bar{x}^2}$	$a = \frac{\bar{y}}{\bar{x}}\left(\frac{\bar{y}(t_1 + t_2) - 2\bar{x} t_1 t_2}{2\bar{y} - \bar{x}(t_1 + t_2)}\right)$
Comments			$(a < 0)$ - ellipse $(a > 0)$ - hyperbola $(a = 0)$ - x - axis parabola $(a = \infty)$ - y - axis parabola

Table (4) Algebraic Fits for Three Parameter Equations (Continued)

$$y = f(x, a, b, c)$$

Name	Cubic	Power Law	Reciprocal Power Law
General Form	$y = ax^3 + bx^2 + cx + d$	$y - k = ax^c + bx$	$y - k = \frac{bx}{ax^c + 1}$
Relative Form	$y = ax^3 + bx^2 + cx$	$y = ax^c + bx$	$y = \frac{bx}{ax^c + 1}$
Tangent Equation	$t = 3ax^2 + 2bx + c$	$t = c\,a\,x^{(c-1)} + b$	$t = \frac{b - c\,y\,a\,x^{(c-1)}}{ax^c + 1}$
"C" Equation	$c = t_1$	$c = \frac{\bar{x}(t_2 - t_1)}{\bar{y} - \bar{x}t_1}$	$c = \frac{t_1(\bar{t} - t_2)}{t_1 - \bar{t}}$
"B" Equation	$b = \frac{3\bar{y} - (2t_1 + t_2)\bar{x}}{\bar{x}^2}$	$b = t_1$	$b = t_1$
"A" Equation	$a = \frac{\bar{x}(t_1 + t_2) - 2\bar{y}}{\bar{x}^3}$	$a = \frac{\bar{y} - \bar{x}t_1}{\bar{x}^c}$	$a = \frac{t_1 - \bar{t}}{\bar{t}\,\bar{x}^c}$
Comments		c must be > 1. For c = 2 this reduces to the y-axis parabola.	c must be > 0